

# Automatic Synthesis of Switching Controllers for Linear Hybrid Automata

## Technical Report

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**Abstract.** In this paper we study the problem of automatically generating switching controllers for the class of Linear Hybrid Automata, with respect to safety objectives. We identify and solve inaccuracies contained in previous characterizations of the problem, providing a sound and complete symbolic fixpoint procedure, based on polyhedral abstractions of the state space. We also prove the termination of each iteration of the procedure. Some promising experimental results are presented, based on an implementation of the fixpoint procedure on top of the tool PHAVer.

## 1 Introduction

Hybrid systems are an established formalism for modeling physical systems which interact with a digital controller. From an abstract point of view, a hybrid system is a dynamic system whose state variables are partitioned into discrete and continuous ones. Typically, continuous variables represent physical quantities like temperature, speed, etc., while discrete ones represent *control modes*, i.e., states of the controller.

Hybrid automata [8] are the most common syntactic variety of hybrid system: a finite set of locations, similar to the states of a finite automaton, represents the value of the discrete variables. The current location, together with the current value of the (continuous) variables, form the instantaneous description of the system. Change of location happens via discrete transitions, and the evolution of the variables is governed by differential equations attached to each location. In a Linear Hybrid Automaton (LHA), the allowed differential equations are in fact differential inclusions of the type  $\dot{\mathbf{x}} \in P$ , where  $\dot{\mathbf{x}}$  is the vector of the first derivatives of all variables and  $P$  is a convex polyhedron. Notice that differential inclusions are non-deterministic, allowing for infinitely many solutions.

The most studied problem for hybrid systems is *reachability*: computing the set of states that are reachable from the initial states, in any amount of time. The reachability problem for LHAs was proved undecidable in [10], indicating that no exact discrete abstraction exists. The complexity standing of the problem was further refined to *semi-decidable* in [14], whose results imply that it is possible

to exactly compute the set of states that are reachable within a bounded number of discrete transitions (*bounded-horizon reachability*).

We study LHAs whose discrete transitions are partitioned into controllable and uncontrollable ones, and we wish to compute a strategy for the controller to satisfy a given goal, regardless of the evolution of the continuous variables and of the uncontrollable transitions. Hence, the problem can be viewed as a *two player game* [13]: on one side the controller, who can only issue controllable transitions, on the other side the environment, who can choose the trajectory of the variables and can take uncontrollable transitions whenever they are enabled.

As control goal, we consider safety, i.e., the objective of keeping the system within a given region of safe states. This problem has been considered several times in the literature. Here, we fix some inaccuracies in previous presentations, propose a sound and complete procedure for the problem<sup>1</sup> and we present a publicly available implementation of the procedure. In particular, we present a novel algorithm for computing the set of states that may reach a given region while avoiding another one, a problem that is at the heart of the synthesis procedure.

Contrary to most recent literature on the subject, we focus on exact algorithms. Although it is established that exact analysis and synthesis of realistic hybrid systems is computationally demanding, we believe that the ongoing research effort on approximate techniques should be based on the solid grounds provided by the exact approach. For instance, a tool implementing an exact algorithm (like our PHAVer+) may serve as a benchmark to evaluate the performance and the precision of an approximate tool.

*Related work.* The idea of automatically synthesizing controllers for dynamic systems arose in connection with discrete systems [12]. Then, the same idea was applied to real-time systems modeled by timed automata [11], thus coming one step closer to the continuous systems that control theory usually deals with. Finally, it was the turn of hybrid systems [14,9], and in particular of Linear Hybrid Automata, the very model that we analyze in this paper. Wong-Toi proposed the first symbolic semi-procedure to compute the controllable region of a LHA w.r.t. a safety goal [14]. The heart of the procedure lies in the operator  $flow\_avoid(U, V)$ , which computes the set of system configurations from which a continuous trajectory may reach the set  $U$  while avoiding the set  $V$  (hence, in this paper we call this operator *RWA*, for *Reach While Avoiding*). Tomlin et al. [13] and Balluchi et al. [3] analyze much more expressive models, with generality in mind rather than automatic synthesis. Their *Reach* and *Unavoid\_Pre* operators, respectively, again correspond to  $flow\_avoid$ .

As explained in Section 3.4, the algorithm provided in [14] for  $flow\_avoid$  does not work for non-convex  $V$ , a case which is very likely to occur in practice, even if the original safety goal is convex. A slightly different algorithm for  $flow\_avoid$  is

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<sup>1</sup> In other words, an algorithm that may or may not terminate, and that provides the correct answer whenever it terminates.

reported to have been implemented in the tool HONEYTECH [6], and we compare it with ours in Section 3.4.

Asarin et al. [1] investigate the synthesis problem for hybrid systems where all discrete transitions are controllable and the trajectories satisfy given linear differential equations of the type  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ . The expressive power of these constraints is incomparable with the one offered by the differential inclusions occurring in LHAs. In particular, linear differential equations give rise to deterministic trajectories, while differential inclusions are non-deterministic. In control theory terms, differential inclusions can represent the presence of environmental *disturbances*. The tool *d/dt* [2], by the same authors, is reported to support controller synthesis for safety objectives, but the publicly available version in fact does not.

The rest of the paper is organized as follows. Section 2 introduces and motivates the model. In Section 3, we present the semi-procedure which solves the synthesis problem. Section 4 reports some experiments performed on our implementation of the procedure, while Section 5 draws some conclusions.

## 2 Linear Hybrid Automata

A *convex polyhedron* is a subset of  $\mathbb{R}^n$  that is the intersection of a finite number of strict and non-strict affine half-spaces. A *polyhedron* is a subset of  $\mathbb{R}^n$  that is the union of a finite number of convex polyhedra. For a general (i.e., not necessarily convex) polyhedron  $G \subseteq \mathbb{R}^n$ , we denote by  $cl(G)$  its topological closure, and by  $\llbracket G \rrbracket \subseteq 2^{\mathbb{R}^n}$  its representation as a finite set of convex polyhedra.

Given an ordered set  $X = \{x_1, \dots, x_n\}$  of variables, a *valuation* is a function  $v : X \rightarrow \mathbb{R}$ . Let  $Val(X)$  denote the set of valuations over  $X$ . There is an obvious bijection between  $Val(X)$  and  $\mathbb{R}^n$ , allowing us to extend the notion of (convex) polyhedron to sets of valuations. We denote by  $CPoly(X)$  (resp.,  $Poly(X)$ ) the set of convex polyhedra (resp., polyhedra) on  $X$ .

We use  $\dot{X}$  to denote the set  $\{\dot{x}_1, \dots, \dot{x}_n\}$  of dotted variables, used to represent the first derivatives, and  $X'$  to denote the set  $\{x'_1, \dots, x'_n\}$  of primed variables, used to represent the new values of variables after a transition. Arithmetic operations on valuations are defined in the straightforward way. An *activity* over  $X$  is a differentiable function  $f : \mathbb{R}^{\geq 0} \rightarrow Val(X)$ . Let  $Acts(X)$  denote the set of activities over  $X$ . The *derivative*  $\dot{f}$  of an activity  $f$  is defined in the standard way and it is an activity over  $\dot{X}$ . A *Linear Hybrid Automaton*  $H = (Loc, X, Edg_c, Edg_u, Flow, Inv, Init)$  consists of the following:

- A finite set  $Loc$  of *locations*.
- A finite set  $X = \{x_1, \dots, x_n\}$  of continuous, real-valued *variables*. A *state* is a pair  $\langle l, v \rangle$  of a location  $l$  and a valuation  $v \in Val(X)$ .
- Two sets  $Edg_c$  and  $Edg_u$  of *controllable* and *uncontrollable transitions*, respectively. They describe instantaneous changes of locations, in the course of which variables may change their value. Each transition  $(l, \mu, l') \in Edg_c \cup Edg_u$  consists of a *source location*  $l$ , a *target location*  $l'$ , and a *jump relation*  $\mu \in Poly(X \cup X')$ , that specifies how the variables may change their value

during the transition. The projection of  $\mu$  on  $X$  describes the valuations for which the transition is enabled; this is often referred to as a *guard*.

- A mapping  $Flow : Loc \rightarrow CPoly(\dot{X})$  attributes to each location a set of valuations over the first derivatives of the variables, which determines how variables can change over time.
- A mapping  $Inv : Loc \rightarrow Poly(X)$ , called the *invariant*.
- A mapping  $Init : Loc \rightarrow Poly(X)$ , contained in the invariant, defining the *initial states* of the automaton.

We use the abbreviations  $S = Loc \times Val(X)$  for the set of states and  $Edg = Edg_c \cup Edg_u$  for the set of all transitions. Moreover, we let  $InvS = \bigcup_{l \in Loc} \{l\} \times Inv(l)$  and  $InitS = \bigcup_{l \in Loc} \{l\} \times Init(l)$ . Notice that  $InvS$  and  $InitS$  are sets of states. Given a set of states  $A$  and a location  $l$ , we denote by  $A \downarrow_l$  the projection of  $A$  on  $l$ , i.e.  $\{v \in Val(X) \mid \langle l, v \rangle \in A\}$ .

## 2.1 Semantics

The behavior of a LHA is based on two types of transitions: *discrete* transitions correspond to the  $Edg$  component, and produce an instantaneous change in both the location and the variable valuation; *timed* transitions describe the change of the variables over time in accordance with the  $Flow$  component.

Given a state  $s = \langle l, v \rangle$ , we set  $loc(s) = l$  and  $val(s) = v$ . An activity  $f \in Acts(X)$  is called *admissible from  $s$*  if (i)  $f(0) = v$  and (ii) for all  $\delta \geq 0$  it holds  $f(\delta) \in Flow(l)$ . We denote by  $Adm(s)$  the set of activities that are admissible from  $s$ . Additionally, for  $f \in Adm(s)$ , the *span* of  $f$  in  $l$ , denoted by  $span(f, l)$  is the set of all values  $\delta \geq 0$  such that  $\langle l, f(\delta') \rangle \in InvS$  for all  $0 \leq \delta' \leq \delta$ . Intuitively,  $\delta$  is in the span of  $f$  iff  $f$  never leaves the invariant in the first  $\delta$  time units. If all non-negative reals belong to  $span(f, l)$ , we write  $\infty \in span(f, l)$ .

*Runs.* Given two states  $s, s'$ , and a transition  $e \in Edg$ , there is a *discrete step*  $s \xrightarrow{e} s'$  with *source*  $s$  and *target*  $s'$  iff (i)  $s, s' \in InvS$ , (ii)  $e = (loc(s), \mu, loc(s'))$ , and (iii)  $(val(s), val(s')[X'/X]) \in \mu$ , where  $val(s')[X'/X]$  is the valuation in  $Val(X')$  obtained from  $s'$  by renaming each variable in  $X$  with the corresponding primed variable in  $X'$ . There is a *timed step*  $s \xrightarrow{\delta, f} s'$  with *duration*  $\delta \in \mathbb{R}^{\geq 0}$  and activity  $f \in Adm(s)$  iff (i)  $s \in InvS$ , (ii)  $\delta \in span(f, loc(s))$ , and (iii)  $s' = \langle loc(s), f(\delta) \rangle$ . For technical convenience, we admit timed steps of duration zero<sup>2</sup>. A special timed step is denoted  $s \xrightarrow{\infty, f}$  and represents the case when the system follows an activity forever. This is only allowed if  $\infty \in span(f, loc(s))$ . Finally, a *joint step*  $s \xrightarrow{\delta, f, e} s'$  represents the timed step  $s \xrightarrow{\delta, f} \langle loc(s), f(\delta) \rangle$  followed by the discrete step  $\langle loc(s), f(\delta) \rangle \xrightarrow{e} s'$ .

<sup>2</sup> Timed steps of duration zero can be disabled by adding a clock variable  $t$  to the automaton and requesting that each discrete transition happens when  $t > 0$  and resets  $t$  to 0 when taken.

A *run* is a sequence

$$r = s_0 \xrightarrow{\delta_0, f_0} s'_0 \xrightarrow{e_0} s_1 \xrightarrow{\delta_1, f_1} s'_1 \xrightarrow{e_1} s_2 \dots s_n \dots \quad (1)$$

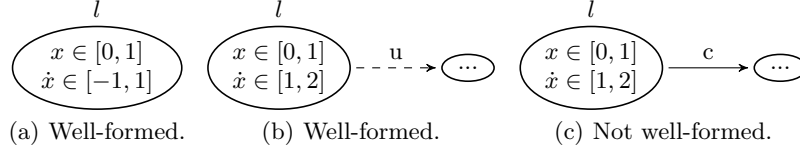
of alternating timed and discrete transitions, such that either the sequence is infinite, or it ends with a timed transition of the type  $s_n \xrightarrow{\infty, f}$ . If the run  $r$  is finite, we define  $\text{len}(r) = n$  to be the length of the run, otherwise we set  $\text{len}(r) = \infty$ . The above run is *non-Zeno* if for all  $\delta \geq 0$  there exists  $i \geq 0$  such that  $\sum_{j=0}^i \delta_j > \delta$ . We denote by  $\text{States}(r)$  the set of all states visited by  $r$ . Formally,  $\text{States}(r)$  is the smallest set containing all states  $\langle \text{loc}(s_i), f_i(\delta) \rangle$ , for all  $0 \leq i \leq \text{len}(r)$  and all  $0 \leq \delta \leq \delta_i$ . Notice that the states from which discrete transitions start (states  $s'_i$  in (1)) appear in  $\text{States}(r)$ . Moreover, if  $r$  contains a sequence of one or more zero-time timed transitions, all intervening states appear in  $\text{States}(r)$ .

*Zenoness and well-formedness.* A well-known problem of real-time and hybrid systems is that definitions like the above admit runs that take infinitely many discrete transitions in a finite amount of time (i.e., *Zeno* runs), even if such behaviors are physically meaningless. In this paper, we assume that the hybrid automaton under consideration generates no such runs. This is easily achieved by using an extra variable, representing a clock, to ensure that the delay between any two transitions is bounded from below by a constant. We leave it to future work to combine our results with more sophisticated approaches to Zenoness known in the literature [3,5].

Moreover, we assume that the hybrid automaton under consideration is *non-blocking*, i.e., whenever the automaton is about to leave the invariant there must be an uncontrollable transition enabled. Formally, for all states  $s$  in the invariant, if all activities  $f \in \text{Adm}(s)$  eventually leave the invariant, there exists one such activity  $f$  and a time  $\delta \in \text{span}(f, \text{loc}(s))$  such that  $s' = \langle \text{loc}(s), f(\delta) \rangle$  is in the invariant and there is an uncontrollable transition  $e \in \text{Edg}_u$  such that  $s' \xrightarrow{e} s''$ . If a hybrid automaton is non-Zeno and non-blocking, we say that it is *well-formed*. In the following, all hybrid automata are assumed to be well-formed.

*Example 1.* Consider the LHAs in Figure 1. The fragment in Figure 1(a) is well-formed, because the system may choose derivative  $\dot{x} = 0$  and remain indefinitely in location  $l$ . The fragment in Figure 1(b) is also well-formed, because the system cannot remain in  $l$  forever, but an uncontrollable transition leading outside is always enabled. Finally, the fragment in Figure 1(c) is not well-formed, because the system cannot remain in  $l$  forever, and no uncontrollable transition is enabled.

*Strategies.* A *strategy* is a function  $\sigma : S \rightarrow 2^{\text{Edg}_c \cup \{\perp\}} \setminus \emptyset$ , where  $\perp$  denotes the null action. Notice that our strategies are *non-deterministic* and *memoryless* (or *positional*). A strategy can only choose a transition which is allowed by the automaton. Formally, for all  $s \in S$ , if  $e \in \sigma(s) \cap \text{Edg}_c$ , then there exists



**Fig. 1.** Three LHA fragments. Locations contain the invariant (first line) and the flow constraint (second line). Solid (resp., dashed) edges represent controllable (resp., uncontrollable) transitions. Guards are **true**.

$s' \in S$  such that  $s \xrightarrow{e} s'$ . Moreover, when the strategy chooses the null action, it should continue to do so for a positive amount of time, along each activity that remains in the invariant. If all activities immediately exit the invariant, the above condition is vacuously satisfied. Formally, if  $\perp \in \sigma(s)$ , for all  $f \in \text{Adm}(s)$  there exists  $\delta > 0$  such that for all  $0 < \delta' < \delta$  it holds  $\delta' \notin \text{span}(f, \text{loc}(s))$  or  $\perp \in \sigma(\langle \text{loc}(s), f(\delta') \rangle)$ . This ensures that the null action is enabled in right-open regions, so that there is an earliest instant in which a controllable transition becomes mandatory.

Notice that a strategy can always choose the null action. The well-formedness condition ensures that the system can always evolve in some way, be it a timed step or an uncontrollable transition. In particular, even if we are on the boundary of the invariant we allow the controller to choose the null action, because, in our interpretation, it is not the responsibility of the controller to ensure that the invariant is not violated.

We say that a run like (1) is *consistent* with a strategy  $\sigma$  if for all  $0 \leq i < \text{len}(r)$  the following conditions hold:

- for all  $\delta \geq 0$  such that  $\sum_{j=0}^{i-1} \delta_j \leq \delta < \sum_{j=0}^i \delta_j$ , we have  $\perp \in \sigma(r(\delta))$ ;
- if  $e_i \in \text{Edg}_c$  then  $e_i \in \sigma(s'_i)$ .

We denote by  $\text{Runs}(s, \sigma)$  the set of runs starting from the state  $s$  and consistent with the strategy  $\sigma$ . The following result ensures that each strategy has at least one run that is consistent with it, otherwise the controller may surreptitiously satisfy the safety objective by blocking the system. The result can be proved by induction by considering that: as long as the strategy chooses the null action, the system may continue along one of the activities that remain within the invariant; if a state is reached from which all activities immediately leave the invariant, the well-formedness assumption ensures that there exists an uncontrollable transition that is enabled; finally, if the strategy chooses a discrete transition, that transition is enabled.

**Theorem 1.** *Given a well-formed hybrid automaton, for all strategies  $\sigma$  and initial states  $s \in \text{InitS}$ , there exists a run that starts from  $s$  and is consistent with  $\sigma$ .*

*Safety control problem.* Given a hybrid automaton  $H$  and a set of states  $T \subseteq \text{InvS}$ , the *safety control problem* asks whether there exists a strategy  $\sigma$  such that, for all initial states  $s \in \text{InitS}$ , all runs  $r \in \text{Runs}(s, \sigma)$  it holds  $\text{States}(r) \subseteq T$ . We call the above  $\sigma$  a *winning strategy*.

### 3 Safety Control

In this section, we consider a fixed hybrid automaton and we present a sound and complete procedure to solve the safety control problem.

#### 3.1 The Abstract Algorithm

We start by defining some preliminary operators. For a set of states  $A$  and  $x \in \{u, c\}$ , let  $\text{Pre}_x^m(A)$  (for *may predecessors*) be the set of states where some discrete transition belonging to  $\text{Edg}_x$  is enabled, which leads to  $A$ , and let  $\bar{A}$  be the set complement of  $A$ . Analogously, let  $\text{Pre}_x^M(A) = \text{Pre}_x^m(A) \setminus \text{Pre}_x^m(\bar{A})$  (the *must predecessors*) be the set of states where all enabled discrete transitions belonging to  $\text{Edg}_x$  lead to  $A$ , and there is at least one such transition enabled.

**Theorem 2.** *The answer to the safety control problem for safe set  $T \subseteq \text{InvS}$  is positive if and only if*

$$\text{InitS} \subseteq \nu W . T \cap \text{CPre}(W),$$

where  $\text{CPre}$  is the controllable predecessor operator below.

*Controllable predecessor operator.* For a set of states  $A$ , the operator  $\text{CPre}(A)$  returns the set of states from which the controller can ensure that the system remains in  $A$  during the next joint transition. This happens if for all activities chosen by the environment and all delays  $\delta$ , one of two situations occurs:

- either the system stays in  $A$  up to time  $\delta$ , while all uncontrollable transitions enabled up to time  $\delta$  (included) also lead to  $A$ , or
- there exists a time  $\delta' < \delta$ , such that the system stays in  $A$  up to time  $\delta'$ , all uncontrollable transitions enabled up to time  $\delta'$  (included) also lead to  $A$ , and the controller can issue a transition at time  $\delta'$  leading to  $A$ .

To improve readability, for a set of states  $A$ , an activity  $f$ , and a time delay  $\delta \geq 0$  (including infinity), we denote by  $\text{While}(A, f, \delta)$  the set of states from where following the activity  $f$  for  $\delta$  time units keeps the system in  $A$  all the time, and any uncontrollable transition taken meanwhile also leads into  $A$ . Formally,

$$\text{While}(A, f, \delta) = \left\{ s \in S \mid \forall 0 \leq \delta' \leq \delta : \langle \text{loc}(s), f(\delta') \rangle \in A \setminus \text{Pre}_u^m(\bar{A}) \right\}.$$

We can now formally define the  $\text{CPre}$  operator and prove Theorem 2.

$$\begin{aligned} \text{CPre}(A) = \left\{ s \in S \mid \forall f \in \text{Adm}(s), \delta \in \text{span}(f, \text{loc}(s)) : s \in \text{While}(A, f, \delta) \right. \\ \left. \text{or } \exists 0 \leq \delta' < \delta : s \in \text{While}(A, f, \delta') \text{ and } \langle \text{loc}(s), f(\delta') \rangle \in \text{Pre}_c^m(A) \right\}. \end{aligned}$$

*Proof.* [if] We shall first build a winning strategy in two steps. Let  $W^* = \nu W.T \cap CPre(W)$  and let  $\sigma$  be a strategy defined as follows, for all states  $s$ :

- $\perp \in \sigma(s)$  and
- if  $s \xrightarrow{e} s'$ ,  $s, s' \in W^*$  and  $e \in Edg_c$ , then  $e \in \sigma(s)$ .

While  $\sigma$  is clearly a strategy, it is not necessarily a winning strategy, as it may admit runs which delay controllable actions either beyond the safety set  $W^*$  or beyond their availability. We can, however, recover a winning strategy by restricting  $\sigma$  in appropriate ways. For all states  $s \in S$  and activities  $f \in Adm(s)$ , let

$$D_{f,s} = \{\delta > 0 \mid \forall 0 \leq \delta' \leq \delta : \langle loc(s), f(\delta') \rangle \in W^* \text{ and } \sigma(\langle loc(s), f(\delta') \rangle) \cap Edg_c \neq \emptyset\}.$$

denote the set of positive time units for which the system can follow activity  $f$ , starting from  $s$ , always remaining in  $W^*$  with some controllable transition enabled and available to the controller.

Starting from  $\sigma$ , we can define a new strategy  $\sigma'$  which coincides with  $\sigma$  on all the states, except for the states  $s \in W^*$  with  $Edg_c \cap \sigma(s) \neq \emptyset$ , where it satisfies  $\sigma'(s) \subseteq \sigma(s)$  and the following two conditions:

- a) If there is  $f \in Adm(s)$  such that  $D_{f,s} = \emptyset$ , then  $\perp \notin \sigma'(s)$ ;
- b) for all  $f \in Adm(s)$ , if  $D_{f,s} \neq \emptyset$ , then there exists a  $\delta \in D_{f,s}$  with  $\perp \notin \sigma'(\langle loc(s), f(\delta) \rangle)$  and  $\forall 0 \leq \delta'' < \delta$ ,  $\perp \in \sigma'(\langle loc(s), f(\delta'') \rangle)$ .

Intuitively, the new strategy  $\sigma'$  ensures that following any activity from a state  $s \in W^*$  in which some controllable action is enabled, a controllable action will always be taken before none of them is available and before leaving  $W^*$ .

To prove that  $\sigma'$  is winning, we must show that for every  $s \in InitS$  and every  $r \in Runs(\sigma', s)$ ,  $States(r) \subseteq T$ . Let

$$r = s_0 \xrightarrow{\delta_0, f_0} s'_0 \xrightarrow{e_0} s_1 \xrightarrow{\delta_1, f_1} s'_1 \xrightarrow{e_1} s_2 \dots s_n \dots$$

be a run consistent with  $\sigma'$ . The following properties can be proved:

1. if  $s_i \xrightarrow{\delta_i, f_i} s'_i$  occurs in  $r$ , with  $\delta_i > 0$  and  $s_i \in W^*$ , then for all  $0 \leq \delta' \leq \delta_i$ , it holds  $\langle loc(s_i), f_i(\delta') \rangle \in W^*$ ;
2. if  $s_i \xrightarrow{\infty, f_i}$  occurs in  $r$  and  $s_i \in W^*$ , then for all  $\delta' \geq 0$ , it holds  $\langle loc(s_i), f_i(\delta') \rangle \in W^*$ ;
3. if  $s_i \xrightarrow{e} s'_i$  occurs in  $r$  and  $s_i \in W^*$ , then  $s'_i \in W^*$ .

We shall prove property (1), as (2) can be proved similarly. Since  $\delta_i > 0$ , by the consistency of  $r$  with  $\sigma'$ , we have  $\perp \in \sigma'(s_i)$ . Assume, by contradiction, that  $\langle loc(s_i), f_i(\delta') \rangle \notin W^*$  for some  $0 < \delta' < \delta_i$ . Since  $s_i \in W^* = CPre(W^*)$ , then  $s_i \in While(W^*, f_i, \delta)$  for some  $\delta \in \mathbb{R}^{\geq 0} \cup \{\infty\}$ , and either  $\delta = \infty$  or  $s_i \xrightarrow{\delta, f_i} s$  and  $s \in Pre_c^m(W^*)$ .



If  $\delta \geq \delta'$ , we have an immediate contradiction, since it would imply  $s_i \in \text{While}(W^*, f_i, \delta')$  and, therefore,  $\langle \text{loc}(s_i), f_i(\delta') \rangle \in W^*$ .

Assume, then,  $\delta < \delta'$ . Then  $\langle \text{loc}(s_i), f_i(\delta) \rangle \in \text{Pre}_c^m(W^*)$ , i.e.,  $\langle \text{loc}(s_i), f_i(\delta) \rangle \xrightarrow{e} s'$  for some  $e \in \text{Edg}_c$  and  $s' \in W^*$ . Therefore, both  $e \in \sigma'(\langle \text{loc}(s_i), f_i(\delta) \rangle)$  and, by the consistency of  $r$  with  $\sigma'$ ,  $\perp \in \sigma'(\langle \text{loc}(s_i), f_i(\delta) \rangle)$ . Since  $\perp \in \sigma'(\langle \text{loc}(s_i), f_i(\delta) \rangle)$ , by definition of  $\sigma'$  the premise of property *a*) cannot hold. Therefore, by property *b*), there must be a  $\delta \leq \delta^* < \delta'$  with  $\perp \notin \sigma'(\langle \text{loc}(s_i), f_i(\delta^*) \rangle)$ . On the other hand, the consistency of  $r$  requires that  $\perp \in \sigma'(\langle \text{loc}(s_i), f_i(\delta) \rangle)$  for all  $0 \leq \delta < \delta_i$ , which is a contradiction. Therefore, for all  $0 \leq \delta' < \delta_i$ ,  $\langle \text{loc}(s_i), f_i(\delta') \rangle \in W^*$ .

Finally, to prove that  $s'_i \in W^*$  we can proceed again by contradiction, assuming  $s'_i \notin W^*$ . Let  $0 < \delta' < \delta_i$ , then  $\langle \text{loc}(s_i), f_i(\delta') \rangle \in W^*$ . Therefore,  $\langle \text{loc}(s_i), f_i(\delta') \rangle \in \text{CPre}(W^*)$  and there exists  $\delta' \leq \delta^* < \delta_i$  with  $\langle \text{loc}(s_i), f_i(\delta') \rangle \in \text{While}(W^*, f_i, \delta^*)$  and  $\langle \text{loc}(s_i), f_i(\delta^*) \rangle \in \text{Pre}_c^m(W^*)$ . Hence, there is a controllable transition  $e \in \text{Edg}_c$  enabled in  $\langle \text{loc}(s_i), f_i(\delta^*) \rangle$  and leading to  $W^*$ . As a consequence,  $\{e, \perp\} \subseteq \sigma(\langle \text{loc}(s_i), f_i(\delta^*) \rangle)$  and, by condition *b*),  $\perp \notin \sigma'(\langle \text{loc}(s_i), f_i(\delta) \rangle)$ , for some  $\delta^* < \delta < \delta_i$ , which contradicts consistency of  $r$  with  $\sigma'$ , hence  $s'_i \in W^*$ .

Let us consider property (3). We have two cases. If  $e \in \text{Edg}_c$ , then the consistency of  $r$  ensures that  $e \in \sigma'(s_i)$  which, by definition of  $\sigma'$ , requires that  $s_{i+1} \in W^*$ . Assume then that  $e \in \text{Edg}_u$ . Then  $\perp \in \sigma'(s_i)$ . Since  $s_i \in W^* = \text{CPre}(W^*)$ , it must hold  $s_i \in \text{While}(W^*, f, 0)$ , for every  $f \in \text{Adm}(s_i)$ . This, in turn, ensures that  $s_i \in W^* \setminus \text{Pre}_u^m(W^*)$ , therefore, all the uncontrollable transitions enabled in  $s_i$  lead to  $W^*$ . Hence the thesis.

To complete the proof, notice that  $W^* \subseteq T$  and  $s_0 \in \text{Inits} \subseteq W^*$ . An easy induction on the length of  $r$ , using properties (1), (2) and (3), gives the result.

**[only if]** Let  $s \notin W^*$ , we prove that for all strategies there is a run that starts in  $s$ , is consistent with the strategy and leaves  $T$ . Let

- $W_0 = T$ ,
- $W_\alpha = T \cap \text{CPre}(W_{\alpha-1})$ , for a successor ordinal  $\alpha$ , and
- $W_\alpha = \bigcap_{\alpha' < \alpha} W_{\alpha'}$  for a limit ordinal  $\alpha$ .

We proceed by induction on the smallest ordinal  $\lambda$  such that  $s \notin W_\lambda$ . If  $\lambda = 0$ , it holds  $s \notin T$  and the thesis is immediate.

We will show that if  $\lambda > 0$  then  $\lambda$  cannot be a limit ordinal. Assume by contradiction that  $\lambda$  is a limit ordinal. Since  $\lambda$  is the smallest ordinal such that  $s \notin W_\lambda$ , we have  $s \in W_\alpha$ , for all  $\alpha < \lambda$ : this means that  $s \in \bigcap_{\alpha < \lambda} W_\alpha$ . But, since  $\lambda$  is a limit ordinal,  $W_\lambda = \bigcap_{\alpha < \lambda} W_\alpha$  and we have that  $s \in W_\lambda$ , obtaining a contradiction.

Otherwise, if  $\lambda > 0$  is a successor ordinal, we have  $s \in W_{\lambda-1} \setminus W_\lambda$  and  $s \notin \text{CPre}(W_{\lambda-1})$ . According to the definition of  $\text{CPre}$ , there exists an activity  $f \in \text{Adm}(s)$  and  $\delta \in \text{span}(s, f)$  such that  $s \notin \text{While}(W_{\lambda-1}, f, \delta)$  and for all  $0 \leq \delta' < \delta$  either  $s \notin \text{While}(W_{\lambda-1}, f, \delta')$  or  $\langle \text{loc}(s), f(\delta') \rangle \notin \text{Pre}_c^m(W_{\lambda-1})$ .

Let  $\delta^*$  be the infimum of those  $\delta'$  such that  $s \notin \text{While}(W_{\lambda-1}, f, \delta')$ , i.e.,

$$\delta^* = \inf\{\delta \mid s \notin \text{While}(W_{\lambda-1}, f, \delta)\}. \quad (2)$$

Clearly  $0 \leq \delta^* \leq \delta$  and, for all  $0 \leq \bar{\delta} < \delta^*$ ,  $\langle loc(s), f(\bar{\delta}) \rangle \notin Pre_c^m(W_{\lambda-1})$ . Hence, any controllable transition enabled in  $\langle loc(s), f(\bar{\delta}) \rangle$ , for any such  $\bar{\delta}$ , leads outside  $W_{\lambda-1}$ . Therefore, any strategy choosing a controllable transition in some of the states  $\langle loc(s), f(\bar{\delta}) \rangle$  has a consistent run leading outside  $W_{\lambda-1}$ . By inductive hypothesis, we obtain the thesis.

If, on the other hand, the strategy allows the controller to stay inactive in all those states, there is a consistent run that reaches  $\delta^*$ . Then we have two cases. If  $\delta^*$  is in fact the minimum of the above set, according to the definition of *While*, there exists  $\delta_1 < \delta^*$  such that  $\langle loc(s), f(\delta_1) \rangle \in \overline{W_{\lambda-1}} \cup Pre_u^m(\overline{W_{\lambda-1}})$ . Therefore, since the controller may not act before  $\delta^*$  along this strategy, there is a consistent run that reaches  $\langle loc(s), f(\delta_1) \rangle$ , which either is in  $\overline{W_{\lambda-1}}$  or reaches it after an uncontrollable transition. In both cases, the thesis follows from the inductive hypothesis.

Finally, we have the case in which  $\delta^*$  is the infimum but not the minimum of the above set. In this case  $0 \leq \delta^* < \delta$  and  $\langle loc(s), f(\bar{\delta}) \rangle \notin Pre_c^m(W_{\lambda-1})$ , for all  $0 \leq \bar{\delta} \leq \delta^*$ . Consider the choice of  $\sigma$  in state  $\langle loc(s), f(\delta^*) \rangle$ . If  $\perp \notin \sigma(\langle loc(s), f(\delta^*) \rangle)$ , the controller issues a discrete move which leads into  $\overline{W_{\lambda-1}}$ . If, instead,  $\perp \in \sigma(\langle loc(s), f(\delta^*) \rangle)$ , since  $\delta^* < \delta \in span(s, f)$ , by the definition of strategy  $\sigma$  will keep choosing  $\perp$  for a non-zero amount of time  $\gamma$ . By (2), there exists  $\delta^* < \hat{\delta} < \delta^* + \gamma$  such that  $s \notin While(W_{\lambda-1}, f, \hat{\delta})$ . As a consequence, there is a consistent run that reaches a state which either is in  $\overline{W_{\lambda-1}}$  or reaches it after an uncontrollable transition. Once again, the thesis is obtained by inductive hypothesis. ■

### 3.2 Computing the Predecessor Operator on LHAs

In this section, we show how to compute the value of the predecessor operator on a given set of states  $A$ , assuming that the hybrid automaton is a LHA and that we can compute the following operations on arbitrary polyhedra  $G$  and  $G'$ : the Boolean operations  $G \cup G$ ,  $G \cap G$ , and  $\overline{G}$ ; the topological closure  $cl(G)$  of  $G$ ; finally, for a given location  $l \in Loc$ , the *pre-flow* of  $G$  in  $l$ :

$$G \swarrow_l = \{u \in Val(X) \mid \exists \delta \geq 0, c \in Flow(l) : u + \delta \cdot c \in G\}.$$

Notice that, for two convex polyhedra  $P$  and  $P'$ , if  $P \subseteq P'$  then  $P \swarrow_l \subseteq P' \swarrow_l$  (monotonicity), and  $(P \swarrow_l) \swarrow_l = P \swarrow_l$  (idempotence).

In the following, we proceed from the basic components of  $CPre$  to the full operator. Given a set of states  $A$  and a location  $l$ , we denote by  $A \downarrow_l$  the projection of  $A$  on  $l$ , i.e.  $\{v \in Val(X) \mid \langle l, v \rangle \in A\}$ . For all  $A \subseteq InvS$  and  $x \in \{c, u\}$ , it holds:

$$Pre_x^m(A) = InvS \cap \bigcup_{(l, \mu, l') \in Edg_x} \mu^{-1}(A \downarrow_{l'}),$$

where  $\mu^{-1}(Z)$  is the pre-image of  $Z$  w.r.t.  $\mu$ . We also introduce the auxiliary operator  $RWA^m$  (*may reach while avoiding*). Given a location  $l$  and two sets of

variable valuations  $U$  and  $V$ ,  $RWA_l^m(U, V)$  contains the set of valuations from which the continuous evolution of the system *may* reach  $U$  while avoiding  $V \cap \overline{U}$ <sup>3</sup>. Notice that on a dense time domain this is not equivalent to reaching  $U$  while avoiding  $V$ : If an activity avoids  $V$  in a right-closed interval, and then enters  $U \cap V$ , the first property holds, while the latter does not. Formally, we have:

$$RWA_l^m(U, V) = \left\{ u \in \text{Val}(X) \mid \exists f \in \text{Adm}(\langle l, u \rangle), \delta \geq 0 : \right. \\ \left. f(\delta) \in U \text{ and } \forall 0 \leq \delta' < \delta : f(\delta') \in \overline{V} \cup U \right\}.$$

An algorithm for effectively computing  $RWA^m$  is presented in the next section, while the following lemma states the relationship between  $CPre$  and  $RWA^m$ . Intuitively, consider the set  $B_l$  of valuations  $u$  such that from state  $\langle l, u \rangle$  the environment can take a discrete transition leading outside  $A$ , and the set  $C_l$  of valuations  $u$  such that from  $\langle l, u \rangle$  the controller can take a discrete transition into  $A$ . We use the  $RWA^m$  operator to compute the set of valuations from which there exists an activity that either leaves  $A$  or enters  $B_l$ , while staying in the invariant and avoiding  $C_l$ . These valuations do not belong to  $CPre(A)$ , as the environment can violate the safety goal within (at most) one discrete transition. We say that a set of states  $A \subseteq S$  is *polyhedral* if for all  $l \in \text{Loc}$ , the projection  $A|_l$  is a polyhedron.

**Lemma 1.** *For all polyhedral sets of states  $A \subseteq \text{Inv}S$ , we have*

$$CPre(A) = \bigcup_{l \in \text{Loc}} \{l\} \times \left( A|_l \setminus RWA_l^m(\text{Inv}S|_l \cap (\overline{A|_l} \cup B_l), C_l \cup \overline{\text{Inv}S|_l}) \right), \quad (3)$$

where  $B_l = \text{Pre}_u^m(\overline{A})|_l$  and  $C_l = \text{Pre}_c^m(A)|_l$ .

*Proof.* In the following, let  $I_l = \text{Inv}S|_l$ .

[ $\subseteq$ ] Let  $s = \langle l, u \rangle \in CPre(A)$  and let  $f \in \text{Adm}(s)$ . By definition,  $0 \in \text{span}(f, l)$  and hence  $s \in \text{While}(A, f, 0)$ . In particular, this implies that  $s \in A$  and  $u \in A|_l$ .

Assume by contradiction that  $s$  does not belong to the r.h.s. of (3). Since  $u \in A|_l$ , it must be

$$u \in RWA_l^m(I_l \cap (\overline{A|_l} \cup B_l), C_l \cup \overline{I_l}).$$

Then, by definition there exists  $f^* \in \text{Adm}(s)$  and  $\delta^* \geq 0$  such that: (i)  $f^*(\delta^*) \in I_l \cap (\overline{A|_l} \cup B_l)$ , and (ii) for all  $0 \leq \delta < \delta^*$  it holds  $f^*(\delta) \in I_l \cap (\overline{C_l} \cup \overline{A|_l} \cup B_l)$ . In particular, this implies that  $\delta^*$  belongs to  $\text{span}(f^*, l)$ . On the other hand, if we apply the definition of  $CPre(A)$  to the activity  $f^*$ , we obtain that for all  $\delta \in \text{span}(f^*, l)$  either  $s \in \text{While}(A, f^*, \delta)$  or there exists  $\delta' < \delta$  such that  $s \in \text{While}(A, f^*, \delta')$  and  $\langle l, f^*(\delta') \rangle \in \text{Pre}_c^m(A)$ . This implies that either  $f^*(\delta^*) \in A|_l \cap \overline{B_l}$  or there exists  $\delta' < \delta^*$  such that  $f^*(\delta') \in A|_l \cap \overline{B_l} \cap C_l$ , which is a contradiction.

<sup>3</sup> In ATL notation, we have  $RWA_l^m(U, V) \equiv \langle\langle \text{env} \rangle\rangle(\overline{V} \cup U) \mathcal{U} U$ , where *env* is the player representing the environment.

[ $\supseteq$ ] Let  $l \in Loc$  and  $u \in A \upharpoonright_l \setminus RWA_l^m(I_l \cap (\overline{A} \upharpoonright_l \cup B_l), C_l \cup \overline{I}_l)$ . By complementing the definition of  $RWA^m$ , we obtain that for all activities  $f$  that start from  $s = \langle l, u \rangle$  and for all times  $\delta \geq 0$ , either  $f(\delta) \in \overline{I}_l \cup (A \upharpoonright_l \cap \overline{B}_l)$  or there exists  $\delta' < \delta$  such that

$$f(\delta') \in \left( \overline{I}_l \cup (A \upharpoonright_l \cap \overline{B}_l) \right) \cap (C_l \cup \overline{I}_l) = \overline{I}_l \cup (A \upharpoonright_l \cap \overline{B}_l \cap C_l) \triangleq E_l.$$

First, assume that for all  $\delta \geq 0$  it holds  $f(\delta) \in \overline{I}_l \cup (A \upharpoonright_l \cap \overline{B}_l)$ . In this case, for all  $\delta \in span(f, l)$ , the point  $f(\delta)$  belongs to  $A \upharpoonright_l \cap \overline{B}_l$ . In other words,  $s \in While(A, f, \delta)$  and hence  $s \in CPre(A)$ .

Otherwise, there exists  $\delta'$  such that  $f(\delta') \in E_l$ . Let  $\delta^*$  be the infimum of the  $\delta'$  with the above property, i.e.,  $\delta^* = \inf\{\delta' \mid f(\delta') \in E_l\}$ . Notice that it holds  $f(\delta) \in \overline{I}_l \cup (A \upharpoonright_l \cap \overline{B}_l)$  for all  $\delta \leq \delta^*$ , which implies  $s \in While(A, f, \delta^*)$ . If there exists  $\delta \leq \delta^*$  such that  $f(\delta) \in \overline{I}_l$ , again we conclude that for all  $\delta \in span(f, l)$  it holds  $f(\delta) \in A \upharpoonright_l \cap \overline{B}_l$  and hence  $s \in CPre(A)$ . In the rest of the proof, we can assume that  $f(\delta) \in I_l$  for all  $\delta \leq \delta^*$ , and therefore  $\delta^* \in span(f, l)$ .

If  $\delta^*$  is in fact the minimum of the above set, i.e.,  $f(\delta^*) \in E_l$ , then according to the current assumptions we have in particular  $f(\delta^*) \in C_l = Pre_c^m(A) \upharpoonright_l$ . Accordingly,  $s \in CPre(A)$ . Finally, we are left with the case in which  $f(\delta^*) \notin E_l$ . By definition, in any neighbourhood of  $\delta^*$  there is a time  $\delta$  such that  $f(\delta) \in E_l$ . Due to the fact that  $E_l$  is a polyhedron and that  $f$  is differentiable, there exists  $\delta' > \delta^*$  such that  $f(\delta) \in E_l$  for all  $\delta^* < \delta \leq \delta'$ . Therefore,  $s \in While(A, f, \delta')$ , and  $\langle l, f(\delta') \rangle \in C_l = Pre_c^m(A)$ . Again, we obtain that  $s \in CPre(A)$ . ■

### 3.3 Computing the $RWA^m$ operator on LHAs

In this section, we consider a fixed location  $l$ . Given two polyhedra  $G$  and  $G'$ , we define their *boundary* to be

$$bndry(G, G') = (cl(G) \cap G') \cup (G \cap cl(G')).$$

We can compute  $RWA^m$  by the following fixpoint characterization.

**Theorem 3.** *For all locations  $l$  and sets of valuations  $U$ ,  $V$ , and  $W$ , let*

$$\tau(U, V, W) = U \cup \bigcup_{P \in [\overline{V}]} \bigcup_{P' \in [W]} \left( P \cap (bndry(P, P') \cap P' \not\prec_l) \not\prec_l \right). \quad (4)$$

*We have  $RWA_l^m(U, V) = \mu W . \tau(U, V, W)$ .*

Roughly speaking,  $\tau(U, V, W)$  represents the set of points which either belong to  $U$  or do not belong to  $V$  and can reach  $W$  along a straight line which does not cross  $V$ . We can interpret the fixpoint expression  $\mu W . \tau(U, V, W)$  as an incremental refinement of an under-approximation to the desired result. The process starts with the initial approximation  $W_0 = U$ . One can easily verify that  $U \subseteq RWA_l^m(U, V)$ . Additionally, notice that  $RWA_l^m(U, V) \subseteq U \cup \overline{V}$ . The equation refines the under-approximation by identifying its *entry regions*, i.e.,

the boundaries between the area which *may* belong to the result (i.e.,  $\bar{V}$ ), and the area which already belongs to it (i.e.,  $W$ ). That is, let  $P \in \llbracket \bar{V} \rrbracket$  and  $P' \in \llbracket W \rrbracket$ , let  $b = \text{bndry}(P, P')$ , we call  $R = b \cap P' \swarrow_l$  an entry region *from*  $P$  *to*  $P'$ , and also an entry region *of*  $W$ . The set  $R$  contains the points of  $b$  that may reach  $P'$  by following the continuous evolution of the system. Hence, the system may move from  $P$  to  $P'$  through  $R$ . Moreover, the set  $R' = P \cap R \swarrow_l$  contains the points of  $P$  that can move to  $P'$  through  $R$ . Any point in  $\bar{V}$  that may reach an entry region (without reaching  $V$  first) must be added to the under-approximation, since it belongs to  $RWA_l^m(U, V)$ .

*Proof of Theorem 3.* First, we show that the  $\tau$  operator is monotonic w.r.t. its third argument, so that the least fixpoint  $\mu W . \tau(U, V, W)$  is well defined.

**Lemma 2.** *For all polyhedra  $U, V$ , and  $W \subseteq W'$ , it holds  $\tau(U, V, W) \subseteq \tau(U, V, W')$ .*

*Proof.* Assume for simplicity that  $\llbracket W \rrbracket \subseteq \llbracket W' \rrbracket$ . Then, it is sufficient to observe that, for all  $P \in \llbracket \bar{V} \rrbracket$ , the expression  $\bigcup_{P' \in \llbracket W \rrbracket} (P \cap (\text{bndry}(P, P') \cap P' \swarrow_l) \swarrow_l)$  is monotonic w.r.t.  $W$ , since it is composed by monotonic operators. ■

The following lemma allows us to switch from arbitrary activities to piecewise straight lines, within Lemma 4.

**Lemma 3 ([14]).** *For all locations  $l$ , and valuations  $u$  and  $v$ , if there is an activity  $f \in \text{Adm}(\langle l, u \rangle)$  and a time  $\delta \geq 0$  such that  $f(\delta) = v$  avoiding  $V$ , then there is a finite sequence of straightline activities leading from  $u$  to  $v$ , each avoiding  $V$ .*

Theorem 3 is an immediate consequence of the following two lemmas.

**Lemma 4.** *For all locations  $l$  and polyhedra  $U$  and  $V$ , it holds  $RWA_l^m(U, V) \subseteq \mu W . \tau(U, V, W)$ .*

*Proof.* Let  $u \in RWA_l^m(U, V)$  and  $W^* = \mu W . \tau(U, V, W)$ . By definition,  $u \in \bar{V} \cup U$ . If  $u$  belongs to  $U$ , then it belongs to  $W^*$  by definition. If  $u$  belongs to  $\bar{V} \setminus U$ , there must be an activity that starts in  $u$  and reaches a point  $u' \in U$  without visiting  $V \setminus U$ . By Lemma 3, there is a finite sequence of straightline segments leading from  $u$  to  $u'$  and avoiding  $V \setminus U$ . Let  $u_0, u_1, \dots, u_k$  be the corresponding sequence of intermediate corner points, where  $u_0 = u$  and  $u_k = u'$ . We proceed by induction on  $k$ . If  $k = 0$ , it holds  $u = u' \in U$ , and the thesis is trivially true. If  $k > 0$ , we apply the inductive hypothesis to  $u_1$ , and we obtain that  $u_1 \in W^*$ . Consider the straight path from  $u_0 \in \bar{V} \setminus U$  to  $u_1 \in W^*$ . This path crosses into  $W^*$  in a given point  $v$ . Formally,  $v$  is the first point along the path which belongs to  $cl(W^*)$ . Hence, there is at least one convex polyhedron  $P' \in \llbracket W^* \rrbracket$  such that  $v \in cl(P')$ . If there is more than one such polyhedron, pick the one that contains at least one point of the straight path from  $v$  to  $u_1$ . In this way, we have  $v \in P' \swarrow_l$ .

Let  $n$  be the number of convex polyhedra in  $\llbracket \bar{V} \cup U \rrbracket$  that are crossed by the straight path from  $u_0$  to  $v$ . We start a new induction on  $n$ . If  $n = 1$ , the

whole line segment from  $u_0$  to  $v$  is contained in a given  $P \in \llbracket \bar{V} \cup U \rrbracket$ . Hence,  $v \in \text{bdry}(P, P')$ , where  $P'$  is a suitable element of  $\llbracket W^* \rrbracket$ . Summarizing, we have  $v \in \text{bdry}(P, P') \cap P' \swarrow_l$  and  $u_0 \in \{v\} \swarrow_l$ . We conclude that  $u_0 \in W^*$ . If  $n > 1$ , we split the straight path from  $u_0$  to  $v$  into  $n$  segments, defined by the intermediate points  $v_1, \dots, v_{n-1}$ , and we apply the inductive hypothesis to  $v_1$ , obtaining that  $v_1 \in W^*$ . Finally, we use an argument analogous to the one for  $n = 1$  to conclude that  $u_0 \in W^*$ . ■

**Lemma 5.** *For all locations  $l$  and polyhedra  $U$  and  $V$ , it holds  $RWA_l^m(U, V) \supseteq \mu W \cdot \tau(U, V, W)$ .*

*Proof.* It suffices to show that  $RWA_l^m(U, V)$  is a fixpoint of  $r$ , i.e.,  $RWA_l^m(U, V) = \tau(U, V, RWA_l^m(U, V))$ . Let  $u \in \tau(U, V, RWA_l^m(U, V))$ , we shall prove that  $u \in RWA_l^m(U, V)$ . If  $u \in U$ , the thesis is obvious. Otherwise, there exist  $P \in \llbracket \bar{V} \rrbracket$  and  $P' \in \llbracket RWA_l^m(U, V) \rrbracket$  such that  $u \in P \cap (\text{bdry}(P, P') \cap P' \swarrow_l) \swarrow_l$ . Hence, there is a straightline activity  $f \in \text{Adm}(\langle l, u \rangle)$  that reaches a point  $v \in \text{bdry}(P, P') \cap P' \swarrow_l$ , while staying in  $P \subseteq \bar{V}$ . If  $v \in P'$ , we are done, as we have found an activity from  $u$  to  $RWA_l^m(U, V)$  which avoids  $V \setminus U$ . Otherwise,  $v$  belongs to  $\text{cl}(P') \cap P' \swarrow_l$  and, therefore, can reach some point  $x \in P'$  through an arbitrarily small flow step along some activity. Since  $P' \subseteq RWA_l^m(U, V)$ , any other possible point  $z$  between  $v$  and  $x$  along the activity belongs to  $P'$  and, therefore, cannot belong to  $V \setminus U$ . Hence,  $v \in RWA_l^m(U, V)$  and, consequently, the so does  $u$ .

Finally, let  $u \in RWA_l^m(U, V)$ , we show that  $u \in \tau(U, V, RWA_l^m(U, V))$ . First, notice that  $u \in U \cup \bar{V}$ . If  $u \in U$ , the thesis is obvious. Otherwise, there exist  $P \in \llbracket \bar{V} \rrbracket$  and  $P' \in \llbracket RWA_l^m(U, V) \rrbracket$  such that  $u \in P \cap P'$ . Therefore, we also have  $u \in \text{bdry}(P, P')$  and  $u \in P' \swarrow_l$ . By (4), we obtain the thesis. ■

*Termination.* The following theorem states the termination of the fixpoint procedure defined in Theorem 3.

**Theorem 4.** *The fixpoint procedure for  $RWA^m$  defined in Theorem 3 terminates in a finite number of steps.*

In order to prove Theorem 4, we shall need some additional definitions and notation. Given two polyhedra  $E$  and  $G$  and two convex polyhedra  $P \in \llbracket E \rrbracket$  and  $P' \in \llbracket G \rrbracket$ , if the entry region  $R$  from  $P$  to  $P'$  is not empty, we write  $G \xrightarrow{P, R}_E G'$ , where  $\llbracket G' \rrbracket = \llbracket G \rrbracket \cup \{P \cap R \swarrow_l\}$ , to denote a *refinement step*. The following lemma can easily be proved exploiting idempotence and monotonicity of  $\swarrow_l$ .

**Lemma 6.** *Assume  $G \xrightarrow{P, R}_E G'$ . For all entry regions  $R'$  of  $G'$  that are not entry regions of  $G$  it holds  $R' \subseteq R \swarrow_l$ .*

*Proof.* By definition of entry region,  $R' = \text{bdry}(P', P \cap R \swarrow_l) \cap (P \cap R \swarrow_l) \swarrow_l$ , with  $P' \in E$  and  $P \cap R \swarrow_l \in G$ . Hence, we can write  $R' \subseteq (P \cap R \swarrow_l) \swarrow_l$ . Moreover, from  $(P \cap R \swarrow_l) \subseteq R \swarrow_l$  and by monotonicity and idempotence properties of  $\swarrow_l$  it follows that  $(P \cap R \swarrow_l) \swarrow_l \subseteq R \swarrow_l$ . Hence the thesis  $R' \subseteq (P \cap R \swarrow_l) \swarrow_l \subseteq R \swarrow_l$ . ■

Intuitively, the fixpoint procedure to compute  $RWA^m$  applies, at each iteration  $k \geq 1$ , all the refinement steps of the form  $G \xrightarrow{P,R}_E G'$ , with  $E = \bar{V}$  and  $G = \tau^{k-1}(U, V, U)$  (where  $\tau^0(U, V, U) = U$  and  $\tau^{i+1}(U, V, U) = \tau(U, V, \tau^i(U, V, U))$ ) for every entry region  $R$  of the current under-approximation  $G$ , following a breadth-first policy. The following lemma make the relationship between (sequences of) refinement steps and the  $\tau(\cdot)$  operator precise.

**Lemma 7.** *If  $\pi = G_0 \xrightarrow{P_1, R_1}_E G_1 \xrightarrow{P_2, R_2}_E \dots \xrightarrow{P_m, R_m}_E G_m$  is a sequence of refinement steps with  $R$  entry region of  $G_l$ , then  $R$  is an entry region of  $\tau^m(G_0, \bar{E}, G_0)$ .*

*Proof.* Let  $\bar{\pi} = G_0 \xrightarrow{P_1, R_1}_E G_1 \xrightarrow{P_2, R_2}_E \dots \xrightarrow{P_k, R_k}_E G_k$  be the shortest prefix of  $\pi$  such that  $R$  is entry region of  $G_k$ . Clearly,  $k \leq m$ . We now proceed by induction on  $k$ . If  $k = 0$ , then  $R$  is entry region of  $G_0 = \tau^0(G_0, \bar{E}, G_0)$  and, by monotonicity of the operator  $\tau$ , the thesis holds.

Assume  $k > 0$ . Since  $R$  is entry region of  $G_k$  but not in  $G_{k-1}$  and  $\llbracket G_k \rrbracket = \llbracket G_{k-1} \rrbracket \cup \{P_k \cap R_k \angle l\}$ , it must be  $R = \text{bdry}(P, (P_k \cap R_k \angle l)) \cap (P_k \cap R_k \angle l) \angle l$ , with  $P \in \llbracket E \rrbracket$  and  $R_k$  entry region in  $G_{k-1}$ . By induction hypothesis,  $R_k$  is entry region of  $\tau^{k-1}(G_0, \bar{E}, G_0)$ . Since  $P_k \in \llbracket E \rrbracket$ , by definition of  $\tau$  we have  $(P_k \cap R_k \angle l) \in \tau(G_0, \bar{E}, \tau^{k-1}(G_0, \bar{E}, G_0)) = \tau^k(G_0, \bar{E}, G_0)$ . Therefore,  $R$  is an entry region in  $\tau^k(G_0, \bar{E}, G_0)$ . Again, by monotonicity of  $\tau$ , the thesis follows. ■

We shall now show that the number of different entry regions employed by the fixpoint procedure for  $RWA^m$  is finite and that the number of its iterations is bounded, thus establishing termination of the procedure itself.

We need first some properties of sequences of refinement steps. Given a sequence  $\pi = G_0 \xrightarrow{P_1, R_1}_E G_1 \xrightarrow{P_2, R_2}_E \dots \xrightarrow{P_k, R_k}_E G_k$ ,  $\text{last}(\pi)$  denotes  $G_k$ . Moreover, given a convex polyhedron  $R$ , let  $\text{prune}(\pi, R)$  be the sequence obtained from  $\pi$  by removing all edges which depend on  $R$ , i.e. such that  $R_i \subseteq R \angle l$ . Formally,  $\text{prune}(\pi, R) = G_0 \xrightarrow{P'_1, R'_1}_E G'_1 \xrightarrow{P'_2, R'_2}_E \dots \xrightarrow{P'_m, R'_m}_E G'_m$  is the largest subsequence of  $\pi$  such that  $R'_i \neq R$ , for all  $1 \leq i \leq m$ . Clearly, we have  $m \leq k$ . The following lemma states that  $\text{prune}(\pi, R)$  preserves all the entry regions of  $\text{last}(\pi)$  that do not depend on  $R$ .

**Lemma 8.** *Let  $\pi = G_0 \xrightarrow{P_1, R_1}_E G_1 \xrightarrow{P_2, R_2}_E \dots \xrightarrow{P_k, R_k}_E G_k$  be a sequence of refinement steps and let  $R$  be an entry region of  $G_k$ , such that  $R \not\subseteq R_1 \angle l$ . Then, there exists a subsequence  $\pi'$  of  $\text{prune}(\pi, R_1)$  such that  $R$  is an entry region of  $\text{last}(\pi')$ .*

*Proof.* We proceed by induction on  $k$ . If  $k = 1$ , we have that  $R$  is an entry region of  $G_1$ , with  $R \subseteq R_1 \angle l$ . By Lemma 6  $R$  must be entry region of  $G_0$ .

If  $k > 1$ , let  $j$  be the smallest index such that  $R$  is an entry region in  $G_j$ . If  $j = 0$ , we are done. Otherwise, by Lemma 6 we have  $R \subseteq R_j \angle l$ . Consequently,  $R_j \not\subseteq R_1 \angle l$  (otherwise, by monotonicity it would hold  $R \subseteq R_1 \angle l$ ). Apply the inductive hypothesis to the prefix  $G_0 \xrightarrow{P_1, R_1}_E G_1 \xrightarrow{P_2, R_2}_E \dots \xrightarrow{P_{j-1}, R_{j-1}}_E G_{j-1}$

and to  $R_j$ . We obtain that there exists a sequence  $\pi'$  that starts from  $G_0$ , does not use  $R_1$ , and ends in a polyhedron  $G'$  such that  $R_j$  is an entry region of  $G'$ . Hence, for the sequence  $\pi' \xrightarrow{P_j, R_j}_E G''$  we have that  $R$  is an entry region of  $G''$  and we obtain the thesis. ■

We are now ready to state the main property relating entry regions and sequences of refinement steps.

**Lemma 9.** *Let  $\pi = G_0 \xrightarrow{P_1, R_1}_E G_1 \xrightarrow{P_2, R_2}_E \dots \xrightarrow{P_n, R_n}_E G_n$  be a sequence of refinement steps and let  $R$  be an entry region of  $G_n$ . Then, there exists a subsequence  $\pi' = G_0 \xrightarrow{P'_1, R'_1}_E G'_1 \xrightarrow{P'_2, R'_2}_E \dots \xrightarrow{P'_m, R'_m}_E G'_m$ , such that:  $R$  is an entry region of  $G'_m$  and  $P'_i \neq P'_j$  for all  $1 \leq i < j \leq m$ .*

*Proof.* Let  $\bar{\pi} = G_0 \xrightarrow{P_1, R_1}_E G_1 \xrightarrow{P_2, R_2}_E \dots \xrightarrow{P_k, R_k}_E G_k$  be the shortest prefix of  $\pi$  such that  $R$  is an entry region of  $G_k$ . We proceed by induction on  $k$ . If  $k = 0$  or  $k = 1$ , the thesis immediately follows. If  $k > 1$ , then  $R_k$  is an entry region of  $G_{k-1}$  and  $R$  is an entry region in  $G_k$ . Since  $k$  is the first index for which  $R$  is an entry region in  $G_k$ , we also have  $R \subseteq P_k \cap R_k \not\subseteq_l$ . We can now apply the inductive hypothesis on  $G_0 \xrightarrow{P_1, R_1}_E G_1 \xrightarrow{P_2, R_2}_E \dots \xrightarrow{P_{k-1}, R_{k-1}}_E G_{k-1}$  to obtain the subsequence  $\pi' = G_0 \xrightarrow{P'_1, R'_1}_E G'_1 \xrightarrow{P'_2, R'_2}_E \dots \xrightarrow{P'_h, R'_h}_E G'_h$ , where  $P'_i \neq P'_j$ , for all  $1 \leq i < j \leq h$ , and  $R_k$  is still an entry region of  $G'_h$ .

Hence,  $\pi^* = \pi' \xrightarrow{P_k, R_k}_E G'_{h+1}$  is a sequence of refinement steps, and  $R \subseteq P_k \cap R_k \not\subseteq_l$  implies that  $R$  is an entry region of  $G'_{h+1}$ . Assume  $P'_j = P_k$  for some  $1 \leq j \leq h$ . Considering the subsequence  $\hat{\pi} = G'_{j-1} \xrightarrow{P'_j, R'_j}_E G'_j \xrightarrow{P'_{j+1}, R'_{j+1}}_E \dots \xrightarrow{P'_h, R'_h}_E G'_h$ , two cases may occur:

1. if  $R_k \not\subseteq R'_j \not\subseteq_l$ , then substituting  $\text{prune}(\hat{\pi}, R'_j)$  for  $\hat{\pi}$  in  $\pi^*$  we obtain, by Lemma 8, the desired sequence of refinement steps;
2. if  $R_k \subseteq R'_j \not\subseteq_l$ , then the subsequence  $G_0 \xrightarrow{*}_E G'_j$  of  $\pi'$  is the desired sequence. Indeed, by idempotence of  $\not\subseteq_l$ ,  $R_k \subseteq R'_j \not\subseteq_l$  implies  $R_k \not\subseteq_l \subseteq R'_j \not\subseteq_l$ . Since  $P'_j = P_k$ , we also have that  $P_k \cap R_k \not\subseteq_l \subseteq P'_j \cap R'_j \not\subseteq_l$ . Therefore,  $R \subseteq P_k \cap R_k \not\subseteq_l \subseteq P'_j \cap R'_j \not\subseteq_l$ . Hence,  $R$  is an entry region of  $G'_j$ . ■

An immediate consequence of the previous lemma is that for any entry region  $R$  there is a sequence  $\pi$  of refinement steps discovering  $R$  (i.e. with  $R$  entry region of  $\text{last}(\pi)$ ) whose length is bounded by  $\llbracket E \rrbracket$ .

We can now establish termination of the fixpoint procedure to compute  $RWA^m$ .

*Proof of Theorem 4.* Notice that  $\llbracket \bar{V} \rrbracket$  and  $\llbracket U \rrbracket$  are finite sets of convex polyhedra, therefore so is the number of initial entry regions of  $\llbracket U \rrbracket$ . The fixpoint procedure of Theorem 3 applies the refinement steps in a breadth-first manner starting from the initial entry regions. Therefore, in every iteration each entry region discovered so far is employed in a refinement step. As a consequence of Lemma 9, taking  $E = \bar{V}$  and  $G_0 = U$ , for every entry region there is a sequence of refinement



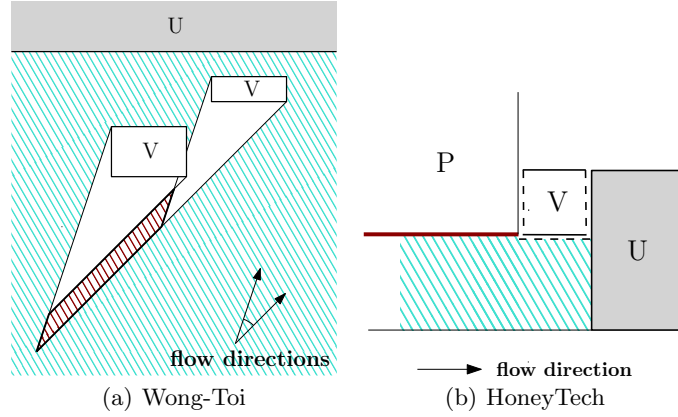
steps discovering it, whose length is bounded by  $|\llbracket \bar{V} \rrbracket|$ . Therefore, by Lemma 7, after at most  $|\llbracket \bar{V} \rrbracket|$  iterations of the procedure all the entry regions have been discovered, and the fixpoint is reached at the next iteration. ■

### 3.4 Previous Algorithms

In the literature, the standard reference for safety control of linear hybrid systems is [14]. The model and the abstract algorithm are essentially similar to ours, except that, differently from our semantics, the states from which a discrete transition is taken are subject to the safety constraint. As to the computation of  $CPre$ , they introduce an operator *flow\_avoid*, which corresponds to our  $RWA^m$  operator. They propose to compute  $RWA_l^m(U, V)$  using the following fixpoint formula:

$$\bigcup_{U' \in \llbracket U \rrbracket} \bigcap_{V' \in \llbracket V \rrbracket} \left( \mu W. U' \cup \bigcup_{P \in \llbracket \bar{V}' \rrbracket} (cl(P) \cap \bar{V}' \cap (W \cap P) \searrow l) \right) \quad (5)$$

A simple example, however, shows that (5) is different from (in particular, larger than)  $RWA_l^m(U, V)$  when  $V$  is non-convex. Consider the example in Figure 2(a), where  $U$  is the gray box on top and  $V$  is the union of the two white boxes. Formula (5) treats the two convex parts of  $V$  separately. As a consequence, the result is the area covered by stripes. However, the correct results should not include the area within the thick border (in red-colored stripes), because any point in that region cannot prevent hitting one of the two convex parts of  $V$ .



**Fig. 2.** Mistakes in previous fixpoint characterizations.

In [6], Deshpande et al. report about an implementation of Wong-Toi’s algorithm in the tool HONEYTECH, obtained as an extension of HyTech. The fixpoint

formula that is meant to capture  $RWA_l^m(U, V)$  is the following:

$$\mu W . U \cup \bigcup_{P \in \llbracket \bar{V} \rrbracket} \left( P \cap (cl(W) \cap cl(P) \cap \bar{V} \cap W \swarrow_l) \swarrow_l \right) \quad (6)$$

Compared to (5), formula (6) correctly treats the case of non-convex  $V$ . However, it suffers from another issue, pertaining the distinction between topologically open and closed polyhedra. Consider the example in Figure 2(b), where  $U$  is the gray box,  $V$  is the white box, and dashed lines represent topologically open sides of polyhedra. The result of applying formula (6) is the area covered by near-vertical stripes. This area includes the thick solid line that starts from a corner of  $V$ . Indeed, if  $W$  is the union of  $U$  and the striped region and  $P \in \llbracket \bar{V} \rrbracket$ , the thick line is exactly  $cl(W) \cap cl(P) \cap \bar{V} \cap W \swarrow_l$ . However, this line does not belong to  $RWA_l^m(U, V)$ , because all its points cannot avoid hitting  $V$  before eventually reaching  $U$ .

## 4 Experiments with PHAVer+

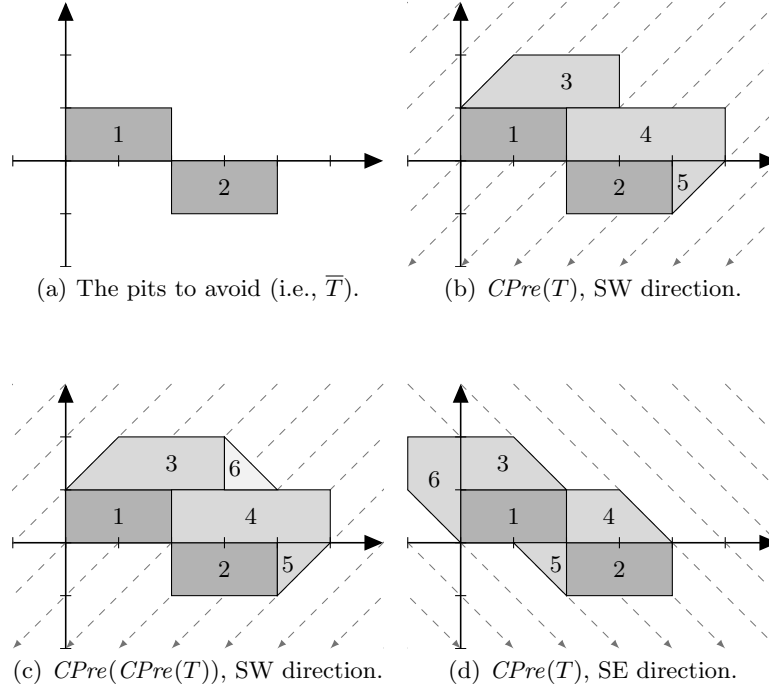
In this section we show experiments about safety control. We implemented the procedure showed in the previous section on the top of the open-source tool PHAVer [7]. The experiments were performed on an Intel Xeon (2.80GHz) PC.

*Truck Navigation Control.* The following example, the Truck Navigation Control (TNC), is derived from the work [6]. Consider an autonomous toy truck, which is responsible for avoiding some 2 by 1 rectangular pits. The truck can take 90-degrees left or right turns: the possible directions are North-East (NE), North-West (NW), South-East (SE) and South-West (SW). One time unit must pass between two changes of direction. The control goal consists in avoiding the pits. Figure 4(a) shows the hybrid automaton that models the system: there is one location for each direction, where the derivative of the position variables ( $x$  and  $y$ ) are set according to the corresponding direction. The variable  $t$  represents a clock ( $\dot{t} = 1$ ) that enforces a one-time-unit wait between turns.

Figure 3 shows the three iterations needed to compute the fixpoint in Theorem 2, in the case of two pits. The safe set is the white area, while the gray region contains the points wherefrom it is not possible to avoid the pits.

The input safe region  $T$  is the area outside the gray boxes 1 and 2 in Figure 3(a). The first iteration (Figure 3(b)) computes  $CPre(T)$  and extends the unsafe set to those points (areas 3, 4, and 5) that will inevitably flow into the pits, before the system reaches  $t = 1$  and the truck can turn. The second iteration (Figure 3(c)) computes  $CPre(CPre(T))$  and extends the unsafe set by adding the area 6: those points may turn before reaching the pits, but after the turn they end up in  $CPre(T)$  anyway (for instance, if turning left, they end up in area 4 of Figure 3(d)). The third iteration reaches the fixpoint.

We tested our implementation on progressively larger versions of the truck model, by increasing the number of pits. We also considered a version of TNC

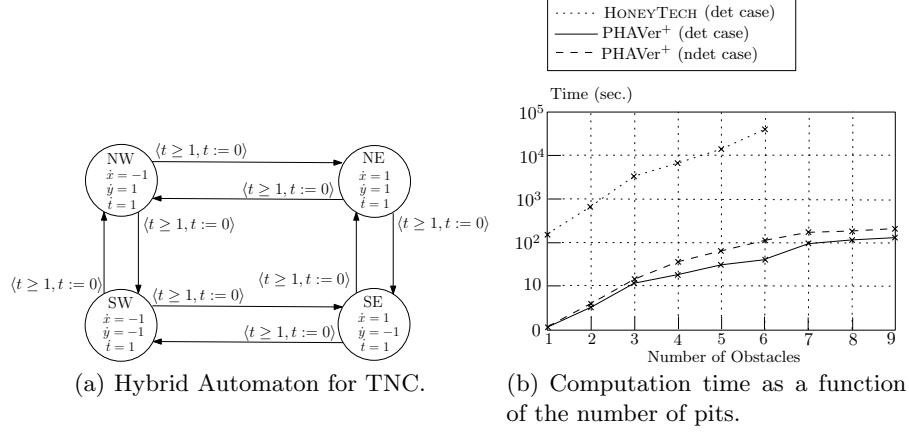


**Fig. 3.** Evolution of the fixpoint in the case of two pits. All figures are cross-sections for  $t = 0$ . Dashed arrows represent flow direction.

with non-deterministic continuous flow, allowing some uncertainty on the exact direction taken by the vehicle. Using an exponential scale, Figure 4(b) compares the performance of our tool (solid line for deterministic model, dashed line for non-deterministic) to the performance reported in [6] (dotted line). We were not able to replicate the experiments in [6], since HONEYTECH is not publicly available.

Because of the different hardware used, only a qualitative comparison can be made: going from 1 to 6 pits (as the case study in [6]), the run time of HONEYTECH shows an exponential behavior, while our tool exhibits an approximately linear growth, as shown in Figure 4(b), where the performance of PHAVer+ is plotted up to 9 pits.

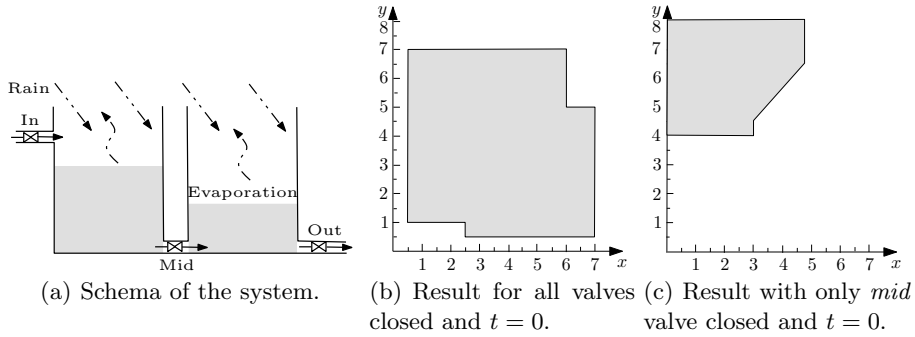
*Water Tank Control.* Consider a system where two tanks — A and B — are linked by a one-directional valve *mid* (from A to B). There are two additional valves: the valve *in* to fill A and the valve *out* to drain B. The two tanks are open-air: the level of the water inside also depends on the potential rain and



**Fig. 4.** Hybrid Automaton and performance for TNC.

evaporation. It is possible to change the state of one valve only after one second since the last valve operation. Figure 5(a) is a schematic view of the system.

The corresponding hybrid automaton has eight locations, one for each combination of the state (open/closed) of the three valves, and three variables:  $x$  and  $y$  for the water level in the tanks, and  $t$  as the clock that enforces a one-time-unit wait between consecutive discrete transitions. Since the tanks are in the same geographic location, rain and evaporation are assumed to have the same rate in both tanks, thus leading to a proper LHA, that is not rectangular [9].



**Fig. 5.** Water Tank Control example.

We set the *in* and *mid* flow rate to 1, the *out* flow rate to 3, the maximum evaporation rate to 0.5 and maximum rain rate to 1, and solve the synthesis problem for the safety specification requiring the water levels to be between 0 and 8. Figure 5(b) (resp., 5(c)) shows the fixpoint result in the case of all valves

closed (resp., *in* and *out* open and *mid* closed). Due to the necessity of one second wait before taking a discrete action, in the case of Figure 5(b),  $x$  and  $y$  must be between 0.5 and 7: otherwise, for example with a level greater than 7 and maximum rain, after one second the level will exceed the limit. In a similar way, with a level less than 0.5 and maximum evaporation, after one second the level would go below the lower bound. The result is computed after 5 iterations in 11 seconds.

## 5 Conclusions

We revisited the problem of automatically synthesizing a switching controller for an LHA w.r.t. safety objectives. The synthesis procedure is based on the  $RWA^m$  operator, for which we presented a novel fixpoint characterization and formally proved its termination.

To the best of our knowledge, this represents the first sound and complete procedure for the task in the literature. We extended the tool PHAVer with our synthesis procedure and performed a series of promising experiments. An account of the challenges involved in the implementation is presented in [4].

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